REAL ANALYSIS
TOPIC 35 - MEASURE (DRAFT)

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1. Extended Real Numbers

Recall that the *extended real numbers* consist of the real numbers together with two symbols for plus and minus infinity:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}.$$

We write ∞ for $+\infty$. We order the set $\overline{\mathbb{R}}$ by defining $-\infty < x < \infty$ for all $x \in \mathbb{R}$. It then makes sense to write $\overline{\mathbb{R}} = [-\infty, \infty]$. When this set is endowed with the order topology, it is homeomorphic to $[0, 1]$.

We will use the following facts regarding extended real numbers without further comment.

- Every nonempty subset of $\overline{\mathbb{R}}$ has a supremum and an infimum in $\overline{\mathbb{R}}$.
- Every monotone sequence in $\overline{\mathbb{R}}$ has a limit in $\overline{\mathbb{R}}$.
- Every series of nonnegative real terms converges in $\overline{\mathbb{R}}$.

Date: April 23, 2019.

2. Set Functions

We are interested in functions which associate an extended real number to each set in a collection of sets. This will allow us to generalize the notion of the length of an interval.

Let X be a set and let $\mathcal{C} \subset X$. A *set function* on $\mathcal{C}$ is a function

$$\gamma : \mathcal{C} \to \overline{\mathbb{R}}.$$

A set function may have one or more of the following properties, assuming that $\mathcal{C}$ is closed under the appropriate unions (finite or countable unions).

- *Monotone*: for $C_1, C_2 \in \mathcal{C}$ with $C_1 \subset C_2$,
$$\gamma(C_1) \leq \gamma(C_2).$$
- *Additive*: for $C_1, C_2 \in \mathcal{C}$ with $C_1 \cap C_2 = \varnothing$,
$$\gamma(C_1 \cup C_2) = \gamma(C_1) + \gamma(C_2).$$
- *Finitely additive*: for $C_1, \dots, C_n \in \mathcal{C}$ with $C_i \cap C_j = \varnothing$ for $i \neq j$,
$$\gamma(\cup_{i=1}^n C_i) = \sum_{i=1}^n \gamma(C_i).$$
- *Countably additive*: for each sequence (C_n) in $\mathcal{C}$ with $C_i \cap C_j = \varnothing$ for $i \neq j$,
$$\gamma(\cup_{i=1}^\infty C_i) = \sum_{i=1}^\infty \gamma(C_i).$$
- *Subadditive*: for $C_1, C_2 \in \mathcal{C}$,
$$\gamma(C_1 \cup C_2) \leq \gamma(C_1) + \gamma(C_2).$$
- *Finitely subadditive*: for $C_1, \dots, C_n \in \mathcal{C}$,
$$\gamma(\cup_{i=1}^n C_i) \leq \sum_{i=1}^n \gamma(C_i).$$
- *Countably additive*: for each sequence (C_n) in $\mathcal{C}$,
$$\gamma(\cup_{i=1}^\infty C_i) \leq \sum_{i=1}^\infty \gamma(C_i).$$

It is clear that additive implies finitely additive, by induction. Also, subadditive implies finitely subadditive. Also, countably additive implies additive, and countably subadditive implies subadditive.

Proposition 1. *Let $\mathcal{A}$ be an algebra of subsets of a set X, and let $\gamma : \mathcal{A} \to [0, \infty]$. If γ is additive, then γ is monotone and subadditive.*

Proof. Suppose that γ is additive.

First we show that γ is monotone. Let $A_1, A_2 \in \mathcal{A}$ with $A_1 \subset A_2$. Then $B = A_2 \smallsetminus A_1 = A_2 \cap A_1^c \in \mathcal{A}$. By additivity, $\gamma(A_2) = \gamma(A_1 \cup B) = \gamma(A_1) + \gamma(B)$, and since $\gamma(B)$ is nonnegative, $\gamma(A_2) \geq \gamma(A_1)$. Thus γ is subadditive.

Next we show that γ is subadditive. Let $A_1, A_2 \in \mathcal{A}$. Let $B = A_1 \cap A_2$. Now $A_1 \cup A_2 = (A_1 \smallsetminus B) \cup B \cup (A_2 \smallsetminus B) = (A_1 \cap B^c) \cup B \cup (A_2 \cap B^C) \in \mathcal{A}$. These sets are disjoint, so $\gamma(A_1 \cup A_2) = \gamma(A_1 \smallsetminus B) + \gamma(B) + \gamma(A_2 \smallsetminus B)$. $\square$

Proposition 2. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $\gamma : \mathcal{A} \to [0, \infty]$. If γ is countably additive, then γ is countably subadditive.*

Proof. Suppose that γ is countably additive. Note that this implies that γ is additive, and hence monotone.

Let (A_n) be a sequence in $\mathcal{A}$. Set $B_1 = A_1$, and for $n \geq 2$, set $B_n = A_n \setminus (\cup_{i=1}^{n-1} A_i)$. Then (B_n) is a sequence of disjoint sets with $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n$. Also, $B_n \subset A_n$ for all n, and by monotonicity, $\gamma(B_n) \leq \gamma(A_n)$. Thus,

$$\gamma(\cup_{n=1}^{\infty} A_n) = \gamma(\cup_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty} \gamma(B_n) \leq \sum_{n=1}^{\infty} \gamma(A_n).$$

□

Proposition 3. *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, and let $\gamma : \mathcal{A} \to [0, \infty]$. If γ is additive and countably subadditive, then γ is countably additive.*

Proof. Suppose that γ is additive and countably subadditive. Then γ is monotone.

Let (A_n) be a sequence of disjoint sets in $\mathcal{A}$. By subadditivity, $\gamma(\cup_{i=1}^{\infty} A_i \leq \sum_{i=1}^{\infty} A_i$, so we wish to see that $\sum_{i=1}^{\infty} A_i \leq \gamma(\cup_{i=1}^{\infty} A_i$.

By additivity and monotonicity, $\sum_{i=1}^{n} \gamma(A_i) = \gamma(\cup_{i=1}^{n} \gamma(A_i) \leq \gamma(\cup_{i=1}^{\infty} \gamma(A_i)$. So, for all n, $\sum_{i=1}^{n} \gamma(A_i)$ is bounded above by $\gamma(\cup_{i=1}^{\infty} \gamma(A_i)$. This inequality survives the limit, so

$$\sum_{i=1}^{\infty} \gamma(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \gamma(A_i) \leq \gamma(\cup_{i=1}^{\infty} \gamma(A_i).$$

□

3. Measures

Definition 1. Let X be a set and let $\mathcal{E}$ be a σ-algebra of subsets of X. A *measure* on $\mathcal{E}$ is a function $\mu : \mathcal{E} \to \overline{\mathbb{R}}$ such that

- **(M1)** $\mu(E) \geq 0$ for all $E \in \mathcal{E}$;
- **(M2)** $\mu(\varnothing) = 0$;
- **(M3)** (E_n) disjoint sequence in $\mathcal{E}$ implies $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

A *measure space* is a triple $(X, \mathcal{E}, \mu)$, where X is a set, $\mathcal{E}$ is a σ-algebra of subsets of X, and μ is a measure on X.

Proposition 4. *Let μ be a measure on a σ-algebra $\mathcal{E}$ of subsets of a set X. Then μ is*

- **(a)** *Additive*
- **(b)** *Subadditive*
- **(c)** *Monotone*
- **(d)** *Countably additive*
- **(e)** *Countably subadditive*

Theorem 1 (Monotone Convergence Theorem)**.** *Let μ be a measure on a σ-algebra $\mathcal{E}$ of subsets of a set X, and let (E_n) be a monotone sequence in $\mathcal{E}$. Then*

(a) *If (E_n) is expanding, then $\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n)$.*

(b) *If (E_n) is contracting, then $\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n)$, provided that there exists a set $A \in \mathcal{E}$ with $E_1 \subset A$ and $\mu(A) < \infty$.*

Proof. **(a)** Suppose (E_n) is expanding. Then $\lim_{n\to\infty} E_n = \cup_{n=1}^{\infty} E_n$.

Set $E_0 = \varnothing$ and for $n \geq 1$, set $F_n = E_n \setminus E_{n-1}$. Since $E_{n-1} \subset E_n$, we have $\mu(F_n) = \mu(E_n) - \mu(E_{n-1})$.

Then (F_n) is a sequence of disjoint sets, and $\cup_{n=1}^{\infty} F_n = \cup_{n=1}^{\infty} E_n$. However, notice that

$$\sum_{i=1}^{n} \mu(F_n) = \sum_{i=1}^{\infty} \mu(E_i) - \mu(E_{i-1}) = \mu(E_n) - \mu(E_0) = \mu(E_n),$$

as this is a telescoping sum. Thus

$$\sum_{n=1}^{\infty} \mu(F_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i) = \lim_{n\to\infty} \mu(E_n).$$

So,

$$\mu(\lim_{n\to\infty} E_n) = \mu(\cup_{n=1}^{\infty} E_n) = \mu(\cup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n) = \lim_{n\to\infty} \mu(E_n).$$

(b) Suppose that (E_n) is contracting. Then $\lim_{n\to\infty} E_n = \cap_{n=1}^{\infty} E_n$.

We assume E_1 is contained in a set of finite measure; let us call it E_0.

Set $F_n = E_n \setminus E_{n+1}$. We claim that

$$\cup_{n=1}^{\infty} F_n = E_1 \setminus \cap_{n=1}^{\infty} E_n.$$

Each F_n is a subset of E_n, and $E_n \subset E_1$. Thus the left hand side is contain in the right hand side. On the other hand, if $x \in E_1 \setminus \cap_{n=1}^{\infty} E_n$, then $x \notin E_{k+1}$ for some k. If we take the smallest such k, we have $x \in F_k = E_k \setminus E_{k+1}$, so $x \in \cup_{n=1}^{\infty} F_n$.

Now (F_n) is a sequence of disjoint sets, and $\mu(F_n) = \mu(E_n) - \mu(E_{n+1})$. So, $\sum_{i=1}^{n} \mu(F_i) = \sum_{i=1}^{n} \mu(E_n) - \mu(E_{n+1}) = \mu(E_1) - \mu(E_{n+1})$, and in the limit,

$$\begin{aligned}
\mu(E_1) - \mu(\cap_{n=1}^{\infty} E_n) &= \mu(E_1 \setminus \cap_{n=1}^{\infty} E_n) \\
&= \mu(\cup_{n=1}^{\infty} F_n) \\
&= \sum_{n=1}^{\infty} \mu(F_n) \\
&= \lim_{n\to\infty} \mu(E_1) - \lim_{n\to\infty} \mu(E_{n+1}) \\
&= \mu(E_1) - \lim_{n\to\infty} \mu(E_n).
\end{aligned}$$

It follows that $\lim_{n\to\infty} \mu(E_n) = \mu(\cap_{n=1}^{\infty})$. $\square$

4. Lengths of Sets

We define what it means to be the length of an interval, of and open sets, and of a bounded closed set. We will use this to define Lebesgue measure in the real numbers, which generalizes length.

Definition 2. Let $A \subset \mathbb{R}$ be nonempty and connected. The *length* of A is

$$\ell(A) = \sup A - \inf A.$$

The types of connected sets are the empty set, singleton sets, and intervals. The length of the empty set is not defined. The length of a singleton is zero. The length of $[a, b)$ is $b - a$.

Recall that an open set is a union of countably many disjoint open intervals. This is what allows the next definition.

Definition 3. Let $G \subset \mathbb{R}$ be open. Define the *length* of G, denoted $\ell(G)$, to be the sum of the length of the disjoint components of G.

Recall that the smallest closed interval of a set A is $\mathrm{sci}(A) = [\inf A, \sup A]$.

Definition 4. Let $F \subset \mathbb{R}$ be bounded and closed, and let $J = \mathrm{sci}(F)$. Define the *length* of F, denoted $\ell(F)$, to be $\ell(J) - \ell(J \setminus F)$.

Proposition 5. *Let $\mathcal{I}$ be a collection of pairwise disjoint subintervals of an interval J. Then the sum of the lengths of the intervals in $\mathcal{I}$ is bounded above by the length of J:*

$$\sum_{I \in \mathcal{I}} \ell(I) \leq \ell(J).$$

Proof. First assume that $\mathcal{I}$ is finite, say $\mathcal{I} = \{I_1, \dots, I_n\}$. Let $a_k = \inf I_k$, and $b_k = \sup I_k$. Let $a = \inf J$ and $b = \sup J$. Then

$$a \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq a_n \leq b_n \leq b.$$

Thus

$$(b - b_n) + (a_n - b_{n-1}) + \cdots + (a_2 - b_1) + (a_1 - a) \geq 0,$$

which implies that $\ell(J) \geq \sum_{k=1}^{n} \ell(I_k)$.

Next, suppose that $\mathcal{I} = \{I_k \mid k \in \mathbb{N}\}$ is an infinite countable collection of intervals. Then for every partial sum, $\sum_{k=1}^{n} \ell(I_k) \leq \ell(J)$. The sequence of partial sums is a bounded nondecreasing sequence, so it converges; thus

$$\sum_{k \in \mathbb{N}} \ell(I_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \ell(I_k) \leq \ell(J).$$

□

Proposition 6. *Let U and V be bounded open sets such that $V \subset U$. Then $\ell(V) \leq \ell(U)$.*

Proof. Each of these open sets is a union of countably many components, which are open intervals. So, let $\{U_\alpha \mid \alpha \in A\}$ be the collection of components of U, where α ranges over some indexing set A. Each component of V is contained in a component of U; let $\{V_{\alpha,\beta} \mid \beta \in B_\alpha\}$ denote the set of components of V which are contained in U_α. Now, by Proposition 5,

$$\ell(V) = \sum_{\alpha \in A} \sum_{\beta \in B_\alpha} \ell(V_{\alpha,\beta}) \leq \sum_{\alpha} \ell(U_\alpha) = \ell(U).$$

□

Corollary 1. *Let U be an open set. If $\mathcal{G}$ is the collection of all bounded open sets containing U, then $\ell(U) = \inf\{\ell(G) \mid G \in \mathcal{G}\}$.*

Proof. First, note that $U \in \mathcal{G}$, so $\ell(U) \geq \inf\{\ell(G) \mid G \in \mathcal{G}\}$. However, if $G \in \mathcal{G}$, then $U \subset G$, so $\ell(U) \leq \ell(G)$ by Proposition 6; thus $\ell(U) \leq \inf\{\ell(G) \mid G \in \mathcal{G}\}$. Thus $\ell(U) = \inf\{\ell(G) \mid G \in \mathcal{G}\}$. □

Lemma 1. *Let $A, B, C \subset \mathbb{R}$ be intervals such that $A = B \cup C$. Then $\ell(A) \leq \ell(B) \cup \ell(C)$.*

Proof. Let $a_1 = \inf A$, $a_2 = \sup A$, $b_1 = \inf B$, $b_2 = \sup B$, $c_1 = \inf C$, and $c_2 = \sup C$. Either $b_1 = a_1$ or $c_1 = a_1$, so, without loss of generality, assume $a_1 = b_1$.

If $C \subset B$, then $\ell(A) = \ell(B) \leq \ell(B) + \ell(C)$, so we may assume that $c_2 = a_2$.

If $b_2 < c_1$, then $\dfrac{b_2 + c_1}{2}$ is in A but not in $B \cup C$. So we see that $b_2 \geq c_1$. Thus

$$\ell(A) = a_2 - a_1 = c_2 - b_1 \leq c_2 - b_1 + (b_2 - c_1) = (c_2 - c_1) + (b_2 - b_1) = \ell(C) + \ell(B).$$

□

Lemma 2. *Let J be a bounded open interval, and let $\mathcal{I}$ be a countable collection of open intervals such that $J = \cup\mathcal{I}$. Then $\ell(J) \leq \sum_{I \in \mathcal{I}} \ell(I)$.*

Proof. Let $a = \inf J$ and $b = \sup J$ so that $J = (a, b)$. For each $\epsilon \in \mathbb{R}$ with $0 < \epsilon < \dfrac{b-a}{2}$, let $F_\epsilon = [a + \epsilon/2, b - \epsilon/2]$. Then $\mathcal{I}$ is an open cover of F_ϵ, and F_ϵ is compact, so $\mathcal{I}$ has a finite subcover, say $\mathcal{C} \subset \mathcal{I}$ with $F_\epsilon \subset \cup\mathcal{C}$.

We claim that $\ell(F_\epsilon) \leq \sum_{C \in \mathcal{C}} \ell(C)$. Since $\mathcal{C}$ is finite, this follows from Lemma 1, and induction.

Now

$$\ell(J) = \ell(F_\epsilon) + \epsilon \leq \sum_{C \in \mathcal{C}} C + \epsilon \leq \sum_{I \in \mathcal{I}} \ell(I) + \epsilon.$$

Since this is true for all $\epsilon > 0$, we have $\ell(J) \leq \sum_{I \in \mathcal{I}} \ell(I)$. □

Lemma 3. *Let U be a bounded open set, and let $\mathcal{I}$ be a countable collection of open intervals such that $U = \cup\mathcal{I}$. Then $\ell(U) \leq \sum_{I \in \mathcal{I}} \ell(I)$.*

Proof. We know that U is the union of countably many disjoint open intervals. Let $\{J_\alpha \mid \alpha \in A\}$ be the collection of components of U, where α ranges over the indexing set A. Then $U = \cup_{\alpha \in A} J_\alpha$.

Each of the intervals in $\mathcal{I}$ is contained in exactly one of the J_α, and the union of the intervals from $\mathcal{I}$ in J_α equals J_α. Let $\{I_{\alpha,\beta} \mid \beta \in B_\alpha\}$ be the collection of intervals from $\mathcal{I}$ which are contained in J_α, where β ranges over the indexing set B_α. Then $J_\alpha = \cup_{\beta \in B_\alpha} I_{\alpha,\beta}$. By Lemma 2,

$$\ell(U) = \sum_{\alpha \in A} \ell(J_\alpha) \leq \sum_{\alpha \in A} \sum_{\beta \in B_\alpha} \ell(I_{\alpha,\beta}) = \sum_{I \in \mathcal{I}} \ell(I).$$

□

Proposition 7. *Let U be a bounded open set, and let $\mathcal{G}$ be a countable collection of open sets such that $U = \cup\mathcal{G}$. Then $\ell(U) \le \sum_{G\in\mathcal{G}} \ell(G)$.*

Proof. Each of the open sets in $\mathcal{G}$ is a union of disjoint open intervals. Let $\mathcal{I}$ be the collection of open intervals in any of the sets in $\mathcal{G}$. Then $\sum_{I\in\mathcal{I}} \ell(I) \le \sum_{G\in\mathcal{G}} \ell(G)$, with equality if and only if no two of the sets in $\mathcal{G}$ share a common component. Thus, by Lemma 3

$$\ell(U) \le \sum_{I\in\mathcal{I}} \ell(I) \le \sum_{G\in\mathcal{G}} \ell(G).$$

$\square$

Proposition 8. *Let F be a bounded closed set contained in an open interval I. Then*

$$\ell(I \setminus F) = \ell(I) - \ell(F).$$

Proof. Let $I = (a, b)$, $c = \inf F$, and $d = \sup F$. Since F is closed and contained in I, $a < c \le d < b$.

Let $G = [c, d]\setminus F$. By definition, $\ell(F) = (d-c)-\ell(G)$, so that $\ell(G) = d-c-\ell(F)$.

Now $I \setminus F$ is the open set which is the disjoint union of G, (a, c), and (d, b). Thus

$$\ell(I\setminus F) = \ell(G)+(c-a)+(b-d) = d-c-\ell(F)+c-a+b-d = b-a-\ell(F) = \ell(I)-\ell(F).$$

$\square$

Proposition 9. *Let F and E be two bounded closed sets such that $F \subset E$. Then $\ell(F) \le \ell(E)$.*

Proof. Let I be an open interval which properly contains E. Let $U = I \setminus F$ and $V = I \setminus E$. By Proposition 8, $\ell(U) = \ell(I) - \ell(F)$ and $\ell(V) = \ell(I) - \ell(E)$.

Since $F \subset E$, we see that $V \subset U$. Thus, by Proposition 6, $\ell(V) \le \ell(U)$; that is, $\ell(I) - \ell(E) \le \ell(I) - \ell(F)$. Thus $\ell(F) \le \ell(E)$. $\square$

Corollary 2. *Let E be a bounded closed set. If $\mathcal{F}$ is the collection of all closed sets contained in E, then $\ell(E) = \sup\{\ell(F) \mid F \in \mathcal{F}\}$.*

Proof. Since $E \in \mathcal{F}$, $\ell(E) \le \sup\{\ell(F) \mid F \in \mathcal{F}\}$. On the other hand, if $F \in \mathcal{F}$, then $F \subset E$, so $\ell(E) \ge \ell(F)$. Thus $\ell(E) \ge \{\ell(F) \mid F \in \mathcal{F}\}$. $\square$

Proposition 10. *Let F be a bounded closed set and let G be a bounded open set, with $F \subset G$. Then $\ell(F) \le \ell(G)$.*

Proof. Let I be an open interval containing G. Then $I = G\cup(I\setminus F)$. By Proposition 7 and Proposition 8,

$$\ell(I) \le \ell(G) + \ell(I \setminus F) = \ell(G) + \ell(I) - \ell(F),$$

whence, $\ell(F) \le \ell(G)$. $\square$

Theorem 2. *Let G be a bounded open set, and let $\mathcal{F}$ be the collection of all closed sets contained in G. Then $\ell(G) = \sup\{\ell(F) \mid F \in \mathcal{F}\}$.*

Proof. If $F \in \mathcal{F}$, then $\ell(G) \geq \ell(F)$ by Proposition 10, so $\ell(G) \geq \sup\{\ell(F) \mid F \in \mathcal{F}\}$.

Let $\epsilon > 0$.

We know that G is a disjoint union of countably many open intervals. If G has finitely many components, let $n \in \mathbb{N}$ denote the number of components, and let $I_1, \ldots, I_n$ be the components. If G has infinitely many components, let $\{I_k \mid k \in \mathbb{N}\}$ be the collection of components; then $\ell(G) = \sum_{i=1}^{\infty} \ell(I_k)$, which converges. Thus there exists $n \in \mathbb{N}$ such that $\ell(G) - \sum_{k=1}^{n} \ell(I_k) < \dfrac{\epsilon}{2}$. Thus $\ell(\cup_{k=n+1}^{\infty} I_k) < \dfrac{\epsilon}{2}$.

Let $I_k = (a_k, b_k)$. Let $\delta_k = \min\{\dfrac{\epsilon}{4n}, \dfrac{b_k - a_k}{3}\}$, and let $F_k = [a_k + \delta_k, b_k - \delta_k]$.+ Let $F = \cup_{k=1} n F_k$. Since F is a union of finitely many closed sets, F is itself closed. Clearly, $F \subset G$, so $F \in \mathcal{F}$.

We see that $\ell(F) = \sum_{k=1}^{n}(b_k - a_k - 2\delta_k) \geq \ell(\cup_{k=1}^{n} I_k) - \dfrac{\epsilon}{2}$. So

$$\ell(G) = \ell(\cup_{k=1}^{n} I_k) + \ell(\cup_{k=n+1}^{\infty} I_k) < \left(\ell(F) + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2},$$

that is, $\ell(G) < \ell(F) + \epsilon$.

So, for every $\epsilon > 0$, there exists $F \in \mathcal{F}$ such that $\ell(G) < \ell(F) + \epsilon$. Thus $\ell(G) \leq \sup\{\ell(F) \mid F \in \mathcal{F}\}$. □

Theorem 3. *Let F be a bounded closed set, and let $\mathcal{G}$ be the collection of all open sets which contain F. Then $\ell(F) = \inf\{\ell(G) \mid G \in \mathcal{G}\}$.*

Proof. If $G \in \mathcal{G}$, then $\ell(F) \leq \ell(G)$ by Proposition 10, so $\ell(F) \leq \inf\{\ell(G) \mid G \in \mathcal{G}\}$.

Let $\epsilon > 0$.

Let I be an open interval which contains F, and set $U = I \setminus F$. Then U is open, and by Theorem 2, $\ell(U)$ is the supremum of the lengths of closed sets contained in U. Thus, there is a closed set $K \subset U$ such that $\ell(U) - \ell(K) < \epsilon$. Let $G = I \setminus K$, so that G is an open set which contains F, and

$$\ell(G) = \ell(I) - \ell(K) < \ell(I) - \ell(U) + \epsilon = \ell(I) - (\ell(I \setminus F) + \epsilon = \ell(F) + \epsilon.$$

Thus, for each $\epsilon > 0$, there exists an open set G containing F such that $\ell(G) < \ell(F) + \epsilon$, which shows that $\ell(F) \geq \inf\{\ell(G) \mid G \in \mathcal{G}\}$. □

5. Lebesgue Measure

Definition 5. Let $A \subset \mathbb{R}$ be bounded.

The *outer measure* of A is

$$\mu^*(A) = \inf\{x \in \overline{\mathbb{R}} \mid x = \ell(G) \text{ for some open set } G \subset \mathbb{R} \text{ such that } A \subset G\}.$$

The *inner measure* of A is

$$\mu_*(A) = \sup\{x \in \overline{\mathbb{R}} \mid x = \ell(K) \text{ for some compact set } K \subset \mathbb{R} \text{ such that } K \subset A\}.$$

We say that A is *measurable* if $\mu^*(A) = \mu_*(A)$. If A is measurable, the *Lebesgue measure* of A is

$$\mu(A) = \mu^*(A) = \mu_*(A).$$

We given a direct proof of the following fact to demonstrate the sort of proof technique we have available. An alternate proof of the following appears later.

Proposition 11. *Let $K \subset \mathbb{R}$ be compact. Then*

$$\ell(K) = \mu^*(K).$$

Proof. Recall that $\ell(K) = \sup K - \inf K - \ell(\operatorname{sci}(K) \smallsetminus K)$.

Let $G = \operatorname{sci}(K) \smallsetminus K$. Then G is a union of countably many disjoint open intervals, which are its components. We will argue the case that G has infinitely many components, the situation for finitely many being a simple change of notation.

Let $G_1, G_2, \dots$ be the components of G. Then there exist real numbers $a_n, b_n \in \mathbb{R}$ such that $G_n = (a_n, b_n)$, for $n \in \mathbb{N}$.

Let $\epsilon > 0$. Set $\delta_n = \min\{\frac{\epsilon}{2^n}, \frac{b_n - a_n}{n}\}$, and let $F_n = [a_n - \delta_n, b_n + \delta_n]$. Note that $\sum_{i=1}^n \delta_i \leq \epsilon \sum_{i=1}^n \frac{1}{2^n} = \epsilon(2 - \frac{1}{2^n})$.

Let $U = (\inf K - \epsilon, \sup K + \epsilon)$. Let $U_n = U \smallsetminus \cup_{i=1}^n F_n$. Then U_n is an open set, and $K \subset U_n$. We have

$$\ell(U_n) = (\sup K - \inf K + 2\epsilon) - \sum_{i=1}^{n}(b_n - a_n + 2\delta_n) \leq \sup K - \inf K + \ell(\cup_{i=1}^n G_i) + 2\epsilon(3 - \frac{1}{2^n}).$$

Take the limit as $n \to \infty$;

$$\mu^*(K) \leq \lim_{n\to\infty} U_n \leq \sup K - \inf K + \ell(G) + 6\epsilon = \ell(K) + 6\epsilon.$$

Since this is true for every $\epsilon > 0$,

$$\mu^*(K) \leq \ell(K).$$

$\square$

Proposition 12. *Let $A \subset \mathbb{R}$ be bounded. Then $\mu_*(A) \leq \mu^*(A)$.*

Proof. Let $\mathcal{F}$ denote the collection of closed sets contained in A, and $\mathcal{G}$ the collection of open sets containing A.

If $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then $F \subset A \subset G$, so $\ell(F) \leq \ell(G)$, by a previous proposition. So,

$$\mu_*(A) = \sup\{\ell(F) \mid F \in \mathcal{F}\} \leq \inf\{\ell(G) \mid G \in \mathcal{G}\} = \mu^*(A).$$

□

Proposition 13 (Measurability of Open Sets)**.** *Let $U \subset \mathbb{R}$ be bounded and open. Then U is measurable, and $\mu(U) = \ell(U)$.*

Proof. Let $\mathcal{G}$ be the collection of open sets that contain U. Then $U \in \mathcal{G}$, so

$$\ell(U) \geq \inf\{\ell(G) \mid G \in \mathcal{G}\}.$$

On the other hand, if G is an open set which contains U, then $\ell(U) \leq \ell(G)$; thus

$$\ell(U) \leq \inf\{\ell(G) \mid G \in \mathcal{G}\}.$$

The result follows. □

Proposition 14 (Measurability of Closed Sets)**.** *Let $K \subset \mathbb{R}$ be bounded and closed. Then K is measurable, and $\mu(K) = \ell(K)$.*

Proof. Let $\mathcal{F}$ be the collection of closed sets contained in K. Then $K \in \mathcal{F}$, so

$$\ell(K) \leq \sup\{\ell(F) \mid F \in \mathcal{F}\}.$$

On the other hand, if F is a closed set which contained in K, then $\ell(K) \geq \ell(F)$; thus

$$\ell(K) \geq \sup\{\ell(F) \mid F \in \mathcal{F}\}.$$

The result follows. □

Proposition 15 (Monotonicity of Outer and Inner Measure)**.** *Let $A, B \subset \mathbb{R}$ be bounded, with $A \subset B$. Then*

(a) $\mu^*(A) \leq \mu^*(B)$
(b) $\mu_*(A) \leq \mu_*(B)$

Proof. We'll discuss **(a)** and then **(b)**, even though they are analogous. We will use the following: if $X \subset Y \subset \mathbb{R}$, then $\inf X \geq \inf Y$, and $\sup X \leq \sup Y$.

(a) Let $\mathcal{U}$ denote the set of all open sets which contains A, and let $\mathcal{V}$ denote the set of all open sets which contain B. Since $A \subset B$, every open set containing B also contains A, so $\mathcal{V} \subset \mathcal{U}$. Therefore, $\{\ell(V) \mid V \in \mathcal{V}\} \subset \{\ell(U) \mid U \in \mathcal{U}\}$. It follows that

$$\mu^*(A) = \inf\{\ell(V) \mid V \in \mathcal{V}\} \leq \inf\{\ell(U) \mid U \in \mathcal{U}\} = \mu^*(B).$$

(b) Let $\mathcal{E}$ denote the set of all closed sets which contained in A, and let $\mathcal{F}$ denote the set of all closed sets which contained B. Since $A \subset B$, every closed set contained in A is also contained in B, so $\mathcal{E} \subset \mathcal{F}$. Therefore, $\{\ell(E) \mid E \in \mathcal{E}\} \subset \{\ell(F) \mid F \in \mathcal{F}\}$. It follows that

$$\mu^*(A) = \sup\{\ell(E) \mid E \in \mathcal{E}\} \leq \sup\{\ell(F) \mid F \in \mathcal{F}\} = \mu^*(B).$$

□

Proposition 16 (Measurability of Sets of Outer Measure Zero)**.** *Let $A \subset \mathbb{R}$ be bounded with $\mu^*(A) = 0$. Then A is measurable, with $\mu(A) = 0$.*

Proof. We know that $0 \leq \mu_*(A) \leq \mu^*(A) = 0$. Thus $\mu_*(A) = 0$, so $\mu^*(A) = \mu_*(A)$, and $A$is measurable, which $\mu(A) = \mu^*(A) = 0$. $\square$

Proposition 17 (Measurability of Countable Sets)**.** *Let $A \subset \mathbb{R}$ be bounded and countable. Then A is measurable, and $\mu(A) = 0$.*

Proof. Let $\epsilon > 0$.

Suppose A is finite, and let $A = \{a_1, \ldots, a_n\}$. Let $U = \cup_{i=1}^n (a_1 - \epsilon/2n, a_i + \epsilon/2n)$. Then $A \subset U$, and $\ell(U) \leq \sum_{i=1}^n \dfrac{\epsilon}{n} = \epsilon$, so $\ell(A) \leq \epsilon$.

Suppose A is infinite, and let A be the image of the sequence (a_i). Let $U = \cup_{i=1}^\infty (a_i - \dfrac{\epsilon}{2 \cdot 2^i}, a_i + \dfrac{\epsilon}{2 \cdot 2^i})$. Then $A \subset U$, and $\ell(U) \leq \sum_{i=1}^\infty \dfrac{\epsilon}{2^n} = \epsilon$, so $\ell(A) \leq \epsilon$.

In either case, $\mu^*(A) \leq \ell(U) \leq \epsilon$, and since ϵ is arbitrary, $\mu^*(A) = 0$. Thus A is measurable, and $\mu(A) = 0$. $\square$

Proposition 18 (Countable Subadditivity of Outer Measure)**.** *Let $\mathcal{C}$ be a countable collection of sets whose union is bounded. Then*

$$\mu^*(\cup \mathcal{C}) \leq \sum_{C \in \mathcal{C}} \mu^*(C).$$

Proof. The result is immediate if the series on the right diverges; thus we assume that it converges.

Let $\mathcal{C} = \{C_1, C_2, \ldots\}$. If $\mathcal{C}$ happens to be finite, let $C_k = \varnothing$ for $k > |\mathcal{C}|$.

For each $i \in \mathbb{N}$ there exists an open set $G_i \supset C_i$ such that $\ell(G_i) < \mu^*(C_i) + \dfrac{\epsilon}{2^i}$, and let $G = \cup_{i=1}^\infty \ell(G_i)$. Then, by subadditivity of lengths of open sets,

$$\ell(G) \leq \sum \ell(G_i) < (\sum_{i=1}^\infty \mu^*(C_i)) + \epsilon.$$

Now $\cup \mathcal{C} \subset G$, so $\mu^*(\cup \mathcal{C}) \leq \ell(G) < \sum_{C \in \mathcal{C}} \mu^*(C) + \epsilon$, and since this is true of all $\epsilon > 0$, the result follows. $\square$

Proposition 19 (Finite Additivity of Measure for Closed Sets)**.** *Let $\mathcal{F}$ be a finite collection of nonempty bounded disjoint closed sets. Then*

$$\mu(\cup\mathcal{F}) = \sum_{F\in\mathcal{F}} \mu(F).$$

Proof. The result is immediate if the series on the right diverges; thus we assume that it converges.

We have shown that for open and closed sets, length equals outer measure equals inner measure. So, we can use these interchangeably, to emphasize our point of view in a given instance.

By Proposition 18, we see that $\mu^*(\cup\mathcal{F}) \leq \sum_{F\in\mathcal{F}} \mu^*(F)$. We wish to show the reverse inequality. It suffices to assume that $\mathcal{F}$ contains two sets, as the finite case will follow by induction.

Let F_1 and F_2 be bounded closed sets. We wish to show that $\mu(F_1 \cup F_2) \geq \mu(F_1) + \mu(F_2)$. Since F_1 and F_2 are compact, there exist disjoint open sets $U_1 \supset F_1$ and $U_2 \supset F_2$.

Let $\epsilon > 0$, and let G be an open set such that $\ell(G) - \mu(F_1 \cup F_2) \leq \epsilon$. Let $G_1 = U_1 \cap G$ and $G_2 = U_2 \cap G$. Then G_1 and G_2 are disjoint open sets contained in G whose union contains $F_1 \cup F_2$, and $\ell(G_1 \cup G_2) \leq \ell(G) \leq \mu^*(F_1 \cup F_2) + \epsilon$. Therefore

$$\mu(F_1) + \mu(F_2) \leq \mu(G_1) + \mu(G_2) \leq \mu(G) \leq \mu^*(F_1 \cup F_2) + \epsilon.$$

Since this is true for all $\epsilon > 0$, we conclude that

$$\mu(F_1) + \mu(F_2) \leq \mu(F_1 \cup F_2).$$

The result follows. $\square$

Proposition 20 (Reverse Subadditivity for Disjoint Closed Sets)**.** *Let $\mathcal{C}$ be a countable collection of disjoint sets whose union is bounded. Then*

$$\mu_*(\cup\mathcal{C}) \geq \sum_{C\in\mathcal{C}} \mu_*(C).$$

Proof. First, suppose that $\mathcal{C}$ is finite. Then $\mathcal{C} = \{C_1, \ldots, C_n\}$ for some distinct disjoint sets C_k. For each k, let $F_k \subset C_k$ be a closed set such that $\mu_*(C_k) - \ell(F_k) < \dfrac{\epsilon}{2n}$. Then $\sum_{k=1}^n \ell(F_k) > \sum_{k=1}^n \mu_*(C_k) + \epsilon$.

Let $F = \cup_{i=1}^n F_k$; then by Proposition 19, $\ell(F) = \sum_{k=1}^n \ell(F_k)$.

Let $C = \cup_{k=1}^n C_k$; then F is a closed set contained in C, so $\ell(F) \leq \mu_*(C)$.

Putting this together,

$$\mu_*(\cup\mathcal{C}) = \mu_*(C) > \ell(F) = \sum_{k=1}^n \ell(F_k) > \sum_{k=1}^n \mu_*(C_k) + \epsilon.$$

Since this is true for every $\epsilon > 0$, the result follows in the finite case.

Now assume that $\mathcal{C}$ is countably infinite, and let $\mathcal{C} = \{C_k \mid k \in \mathbb{N}\}$, where the C_k are distinct. By Proposition 15 and the finite case,

$$\mu_*(\cup_{k=1}^\infty C_k) \geq \mu_*(\cup_{k=1}^n C_k) \geq \sum_{k=1}^n \mu_*(C_k).$$

This is true for all $n \in \mathbb{N}$, so taking the limit as $n \to \infty$,

$$\mu_*(\cup_{k=1}^\infty C_n) \geq \sum_{k=1}^\infty \mu_*(C_k).$$

□

Proposition 21 (Countable Additivity of Measure)**.** *Let $\mathcal{S}$ be a countable collection of disjoint measurable sets. Then $\cup\mathcal{S}$ is measurable, and*

$$\mu(\cup\mathcal{S}) = \sum_{S\in\mathcal{S}} \mu(S).$$

Proof. By Proposition 12, $\mu_*(\cup\mathcal{S}) \le \mu^*(\cup\mathcal{S})$.

By Proposition 18, $\mu^*(\cup\mathcal{S}) \le \sum_{S\in\mathcal{S}} \mu^*(S)$.

By Proposition 20, $\mu_*(\cup\mathcal{S}) \ge \sum_{S\in\mathcal{S}} \mu_*(S)$.

Since the sets in $\mathcal{S}$ are measurable, $\sum_{S\in\mathcal{S}} \mu_*(S) = \sum_{S\in\mathcal{S}} \mu^*(S)$.

Combine these to get

$$\mu^*(\cup\mathcal{S}) \le \sum_{S\in\mathcal{S}} \mu^*(S) = \sum_{S\in\mathcal{S}} \mu_*(S) \le \mu_*(\cup\mathcal{S}) \le \mu^*(\cup\mathcal{S}).$$

The result follows. □

Corollary 3. *If a closed set F is contained in an open set G, then $\mu(G) = \mu(F) + \mu(G \setminus F)$.*

Proof. Since $G\setminus F$ is open, $G\setminus F$ is measurable. The result follows from Proposition 21. □

Corollary 4. *If an open set G is contained in a closed set F, then $\mu(F) = \mu(G) + \mu(F \setminus G)$.*

Proof. Since $F\setminus G$ is open, $F\setminus G$ is measurable. The result follows from Proposition 21. □

Corollary 5. *If F is a closed set contained in a closed interval $[a, b]$, then $[a, b]\setminus F$ is measurable and $\mu([a, b] \setminus F) = (b - a) - \mu(F)$.*

Proof. Let $G = (a, b) \setminus F$. Then G is open, and thus measurable.

If $\{a, b\} \subset F$, then $[a, b] \setminus F = G$ is open, and thus measurable.

Otherwise, if $a \in F$, then $[a, b] \setminus F = G \cup \{b\}$ is the union of measurable sets, and thus measurable.

Otherwise, if $b \in F$, then $[a, b] \setminus F = G \cup \{a\}$ is the union of measurable sets, and thus measurable.

Otherwise, $[a, b] \setminus F = G \cup \{a, b\}$ is the union of measurable sets, and thus measurable.

The result follows from additivity. □

Proposition 22. *Let $A \subset \mathbb{R}$ be bounded. Then A is measurable if and only if for every $\epsilon > 0$ there exists a bounded open set $G \supset A$ and a bounded closed set $F \subset A$ such that $\mu(G) - \mu(F) < \epsilon$.*

Proof. We use the fact that if B is open or closed, we have $\ell(B) = \mu(B)$.

Suppose that A is measurable, and let $\epsilon > 0$. There exists a closed set $F \subset A$ such that $\mu_*(A) - \ell(F) < \dfrac{\epsilon}{2}$. There exists an open set $G \supset A$ such that $\ell(G) - \mu^*(A) < \dfrac{\epsilon}{2}$. Adding these inequalities, and using the fact that $\mu_*(A) = \mu^*(A)$, we get $\mu(G) - \mu(F) < \epsilon$.

Conversely, suppose that for every $\epsilon > 0$ there exists a closed set $F \subset A$ and an open set $G \supset A$ such that $\mu(G) - \mu(F)$. Thus let ϵ, G, and F be as stated. We have We see that $\mu(F) \leq \mu_*(A) \leq \mu^*(A) \leq \mu(G)$. Thus, $\mu_*(A) + \epsilon \geq \mu^*(A)$. Since this is true for all $\epsilon > 0$, we see that $\mu_*(A) \geq \mu^*(A)$. But we always have $\mu_*(A) \leq \mu^*(A)$, so $\mu_*(A) = \mu^*(A)$; therefore, A is measurable. □

Proposition 23. *Let $\mathcal{S}$ be a finite collection of measurable sets. Then $\cup\mathcal{S}$ is measurable.*

Proof. We may assume that $\mathcal{S}$ contains two sets; the finite case follows by induction.

Thus let $\mathcal{S} = \{S_1, S_2\}$, and let $\epsilon > 0$. By Proposition 22, for $k = 1, 2$, there exist closed sets $F_k \subset S_k$ and open sets $G_k \supset S_k$, such that $\mu(G_k) - \mu(F_k) < \dfrac{\epsilon}{2}$. Let $S = S_1 \cup S_2$, $G = G_1 \cup G_2$ and $F = F_1 \cup F_2$. Clearly, $F \subset S \subset G$. Also, $G \smallsetminus F \subset (G_1 \smallsetminus F_1) \cup (G_2 \smallsetminus F_2)$.

Now

$$\mu(G) - \mu(F) = \mu(G \smallsetminus F) \leq \mu(G_1 \smallsetminus F_1) + \mu(G_2 \smallsetminus F_2) = \mu(G_1) - \mu(F_1) + \mu(G_2) - \mu(F_2) < \epsilon.$$

By Proposition 22, S is measurable. □

Proposition 24. *Let $A \subset [a, b]$. Then*

$$\mu^*(A) + \mu_*([a, b] \smallsetminus A) = b - a.$$

Proof. Let $B = [a, b] \smallsetminus A$. We wish to show that $\mu^*(A) + \mu_*(B) = b - a$.

Let $\epsilon > 0$.

There exists an open set $G \supset A$ such that $\ell(G) - \mu^*(A) < \epsilon$. Let $F = [a, b] \smallsetminus G$. Then F is closed, and $F \subset B$. Thus

$$\mu_*(B) \geq \ell(F) \geq b - a - \ell(G) \geq b - a - \mu^*(A) - \epsilon,$$

that is, $\mu^*(A) + \mu_*(B) + \epsilon \geq b - a$. Since this is true for all $\epsilon > 0$, $\mu^*(A) + \mu_*(B) \geq b - a$.

To obtain the reverse inequality, realize that there exists a closed set $F \subset B$ such that $\mu_*(B) - \ell(F) \leq \epsilon$. Clearly $A \subset [a, b] \smallsetminus F$, so by Proposition 5 $\mu_*(A) \leq \mu([a, b] \smallsetminus F) = b - a - \ell(F)$.

Let $\delta > 0$ and let $G = (a - \delta/2, b + \delta/2) \smallsetminus F$. Then G is an open set which contains A, so

$$\mu^*(A) \leq \ell(G) = (b - a + \delta) - \ell(F),$$

whence $\mu^*(A) + \ell(F) \leq b - a + \delta$. Since this is true for all $\delta > 0$, we have $\mu^*(A) + \ell(F) \leq b - a$, whence

$$\mu^*(A) + \mu^*(B) \leq \mu^*(A) + \ell(F) + \epsilon \leq b - a + \epsilon.$$

Since this is true for all $\epsilon > 0$, we have $\mu^*(A) + \mu^*(B) \leq b - a$. □

Corollary 6. *Let $A \subset [a, b]$. Then A is measurable if and only if*

$$\mu^*(A) + \mu^*([a, b] \smallsetminus A) = b - a.$$

Corollary 7. *Let $A \subset [a, b]$. Then A is measurable if and only if $[a, b] \smallsetminus A$ is measurable.*

Proof. We prove both corollaries here.

Let $B = [a, b] \smallsetminus A$. Then $[a, b] \smallsetminus B = A$. Thus, by Proposition 24,

$$\mu^*(A) + \mu_*(B) = b - a \quad \text{and} \quad \mu^*(B) + \mu_*(A) = b - a,$$

so $\mu^*(A) - \mu_*(A) = \mu^*(B) - \mu_*(B)$.

Now A is measurable if and only if the left hand side is zero, which is true if and only if the right hand side is zero, which is true if and only if B is measurable.

Suppose A is measurable. Then so is B, and by Proposition 21, we have

$$\mu^*(A) + \mu^*([a, b] \smallsetminus A) = \mu(A) + \mu(B) = \mu(A \cup B) = \mu([a, b]) = b - a.$$

On the other hand, suppose $\mu^*(A) + \mu^*(B) = b - a$. Combine this Proposition 24 to obtain $\mu_*(B) = \mu^*(B)$, so B is measurable. Hence, A is measurable. □

Proposition 25. *Let I be an open interval and let $A \subset I$. Then $\mu^*(A)+\mu_*(I \smallsetminus A) = \ell(I)$.*

Proof. This is the same as Proposition 24, except the ambient interval is open rather than closed.

Using Proposition 24, we have

$$\mu^*(A) + \mu_*((a, b) \smallsetminus A) \leq \mu^*(A) + \mu_*([b, a] \smallsetminus A) = b - a.$$

Let $C = \{a, b\} \smallsetminus A$. Then $[a, b] \smallsetminus A = ((a, b) \smallsetminus A) \cup C$, and this is a disjoint union. Thus

interval instead of a closed one. □

Proposition 26. *Let $\mathcal{S}$ be a finite collection of bounded measurable sets. Then $\cap_{S \in \mathcal{S}} S$ is measurable.*

Corollary 8. *Let S and T be measurable sets. Then $T \smallsetminus S$ is measurable. Furthermore, if $S \subset T$, then $\mu(T \smallsetminus S) = \mu(T) - \mu(S)$.*

Proposition 27. *Let $\mathcal{S}$ be a countable collection of measurable sets. Then $\cup\mathcal{S}$ is measurable.*

Proposition 28. *Let $\mathcal{S}$ be a countable collection of measurable sets. Then $\cap\mathcal{S}$ is measurable.*

Theorem 4 (Caratheodory Condition)**.** *Let S be a bounded subset of $\mathbb{R}$. Then S is measurable if and only if, for every bounded $A \subset \mathbb{R}$, we have*

$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \cap S^c).$$

Theorem 5. *The collection of all measurable sets is a σ-algebra.*

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